Teaching for Abstraction: Collaborative Teacher Learning

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Working collaboratively with the researchers, a small team of teachers developed and taught a lesson based on the Teaching for Abstraction model (White & Mitchelmore, 2010). This paper reports how one teacher learned about the model and implemented it in her lesson. It was found that she had assimilated several key features of the model, such as starting with several embodiments of the target concept and guiding students to look for similarities between them. However, she had some difficulty focusing students' attention clearly on the underlying mathematical similarity. It was concluded that teachers need to experience more examples of Teaching for Abstraction before they can to reify and apply the model faithfully.

In 1963, Zoltan Dienes advanced a theory that all elementary mathematical concepts are the result of abstraction and generalisation from common experiences. Later, Richard Skemp summarised some key terms in this theory:

> *Abstracting* is an activity by which we become aware of similarities ... among our experiences. *Classifying* means collecting together our experiences on the basis of these similarities. An *abstraction* is some kind of lasting change, the result of abstracting, which enables us to recognise new experiences as having the similarities of an already formed class. ... To distinguish between abstracting as an activity and abstraction as its end-product, we shall ... call the latter a *concept.* (1986, p. 21, italics in original)

Over the past decade, the first and last author have been developing a teaching model, called Teaching for Abstraction, which is based on Dienes's theory. This model (described in more detail in White & Mitchelmore, 2010) consists of four phases:

- *Familiarity*. Students explore a variety of contexts where a concept arises, in order to form generalisations about individual contexts and thus become familiar with the underlying structure of each context.
- *Similarity*. Teaching then focuses on helping students recognise the similarities and differences between the underlying structures of these contexts.
- *Reification*. The general principles underpinning the identified similarities are drawn out, and students are supported to abstract the desired concept into a mental object that can be operated on in its own right.
- *Application*. Students are then directed to new situations where they can use the concept.

In our investigations to date, also described in White and Mitchelmore (2010), we have designed Teaching for Abstraction units on angles, decimals, percentages and ratio and prepared corresponding teaching materials. Our studies of their implementation across Years 3-8 indicate that the model has promise in terms of student learning of key concepts and generalisations but that further exploration and development is still needed.

In particular, it was repeatedly found that some teachers had difficulty understanding and therefore implementing the Teaching for Abstraction model. Some teachers followed the materials provided to the letter, not being in a position to adapt them to their particular

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classroom situation; others reversed the suggested teaching order and thus subverted the essence of the model. A possible reason for this result is that the teachers were not involved in planning the unit or developing the teaching materials. As Pegg and Panizzon (2008) warn, teachers need to be actively involved if innovative ideas are to be accepted.

We therefore decided to investigate a different method of implementing the model, in which we would work with teachers to develop Teaching for Abstraction lessons that would better fit their classroom situations. We hypothesised that such a procedure would lead to improved teacher learning about the model and hence a more faithful implementation.

The Present Study

As a theoretical framework for assessing the impact of this new method, we use the Interconnected Model of Professional Growth (Clarke & Hollingsworth, 2002). This model contains four domains and various interactions between them, as shown in Figure 1. Note that most interactions are bidirectional. For example, teachers enact a new idea, belief or practice in their classroom and then reflection provides (positive or negative) feedback on the critical features of that innovation and its value to them.

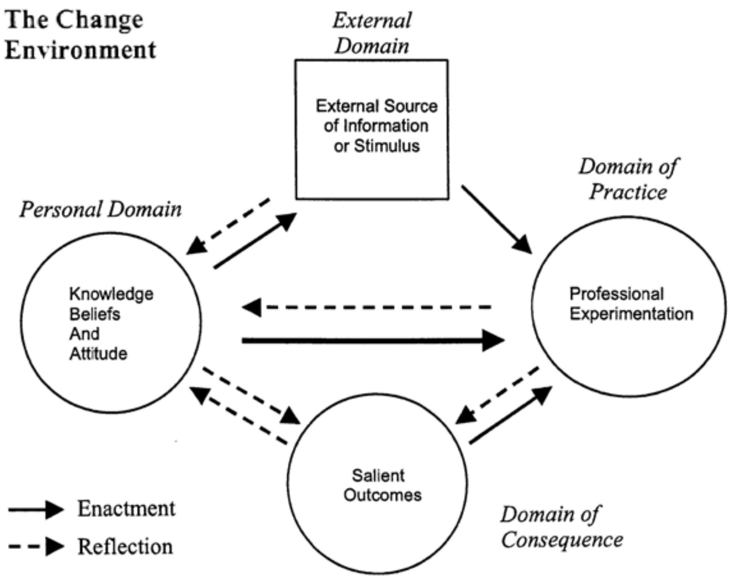

Figure 1. The Interconnected Model of Professional Growth
(from Clarke & Hollingsworth, 2002).

As an initial exploratory study, we worked with a small group of teachers in a small regional town to prepare a number of unconnected lessons. The teachers then taught these lessons in a kind of lesson study mode (Hart, Alston, & Murata, 2011). In terms of Figure 1, the External Domain was the collaborative lesson preparation and study, the Personal Domain was teachers' knowledge and acceptance of Teaching for Abstraction, the Domain of Practice was the lessons taught, and the Salient Outcomes consisted of student outcomes from the lessons taught. Our basic research question was:

- Can collaborative lesson development lead to a faithful implementation of the Teaching for Abstraction model?

To provide answers to this question, data were collected on all four components of the professional growth model in Figure 1. We report a case study of one teacher involved in the collaborative planning and delivery of one Teaching for Abstraction lesson.

Method

Participants

The study took place in Grade 6 at St Joseph's,[1] a small primary school in regional New South Wales. Initially, three teachers volunteered to participate: David, Bridget and Uarda. However, David was unwell during our visits and Bridget had several other calls on her time. Only Uarda participated fully, and she is therefore the focus of this paper. Uarda had been teaching for 5 years, and was enrolled in a master's degree at the time.

Procedures

The authors paid two 2-day visits to St Joseph's. Both visits followed the same pattern: The research team (the authors, Uarda, and whichever of the other teachers were available) met for an initial discussion on the afternoon of Day 1 to decide on the aims for the following day. Early on Day 2, the team developed a lesson and prepared the necessary materials. Later in Day 2, Uarda taught the experimental lesson while the others observed, after which the team discussed the lesson and modified it as they felt appropriate. Bridget then taught the revised lesson while the others observed. Because of space limitations, only the first visit is reported here.

Data Collection and Analysis

The research team's discussion sessions were audio recorded and transcribed, but no recordings were made of the lessons. Instead, one member of the research team acted as a lesson recorder, taking detailed notes of the lessons that included time markers for the major transitions. The other members subsequently added their individual observations.

To assess short-term student learning, the recorder noted students' comments during the face-to-face teaching and all members of the team circulated and observed students during the small group work phase, occasionally interacting with them to clarify what they were attempting to do. In addition, the teachers administered a short quiz at the end of the first visit and a short questionnaire at the end of the second visit.

The analysis of how Uarda interpreted and applied the Teaching for Abstraction model, as well as its resulting effectiveness and potential, focussed on the four components shown in Figure 1. Firstly, each author formed an interpretation of each of these components on the basis of their own observations, notes and informal discussions during the site visits. The three authors then cross-validated and synthesised their separate interpretations during extensive discussions, frequently re-examining transcripts and field notes to reach consensus.

Results

Initial Discussions

At the first meeting of the research team, the authors outlined the Teaching for Abstraction model and gave some examples of their previous research. In particular, they provided the teachers with copies of the instructional materials developed for a previous

[1] We use pseudonyms for the school and the teacher participants.

investigation of teaching percentages in Year 6 (White, Mitchelmore, Wilson, & Faragher, 2008). They also explained the purpose of the study, and asked teachers for their reaction to the model and its potential in their situation.

The teachers expressed interest in experimenting with the model. Uarda indicated that she regularly trialled novel approaches that she believed might be beneficial to her students. Teaching for Abstraction had a definite resonance for her because she was particularly keen on the use of realistic scenarios and always tried to embed mathematics in contexts that she felt would be familiar and interesting to students.

The teachers then outlined a number of topics where they felt their students were having most difficulty, and it was agreed that the next day's lesson would focus on place value in decimals. It was also agreed that Uarda and Bridget would both teach the same lesson with their own classes, with time for discussion and revision between them.

Lesson Planning

The teachers reported that many students were still unable to tell whether, for example, 0.65 was bigger than 0.8. It was decided to focus the lesson on this topic, restricting the content to 1- and 2-place decimals. This topic was considered to be sufficiently narrow for a single lesson that could nevertheless be of significant value to students. The teachers could extend students' understanding to other decimals later.

The team then sought to identify a small number of familiar contexts involving one- or two-place decimals, which could then be compared to identify and abstract the underlying similarity. The teachers initially had some difficulty with this task but, after some suggestions from the authors, the team agreed on four contexts: money ($0.65 vs $0.8), length (0.65 cm vs 0.8 cm), fractions of a box of 100 lollies (0.65 vs 0.8) and fractions of a 10 × 10 square of chocolate (0.65 vs 0.8). It was decided to write four scenarios and to break each class into four groups that would (after an initial introduction) circulate around the four tasks, spending no more than 5 minutes on each. This would leave time for a final discussion in which generalisations about decimal place value could be abstracted.

The teachers then left to collect or create materials for the lesson (counters for money and lollies, rulers and butcher paper for measurement, and Dienes blocks for the chocolate). After they returned, discussion continued on how to phrase the tasks to include a natural comparison. The entire planning session took just under an hour.

The Lesson

Uarda started the lesson by writing the four scenarios on the whiteboard. She explained the rotation procedure for the lesson (which was already familiar practice in her classroom), divided the students into four approximately equally sized groups, and instructed them to start working on their first scenario. No other introduction was given.

During the small group work, the group working on the money task finished first at each rotation. The lollies group took a long time to count out two lots of 100 counters, and the measurement group was slow to draw their two lines. So although the first rotation took only 3 minutes, as planned, the group work took up 26 minutes in all.

Uarda then commenced the final section of the lesson. Asked to identify similarities between the four scenarios, students initially remarked only on superficial aspects: All the scenarios involved Tom and Judy and the numbers 0.8 and 0.65, and Tom always won. Asked how we know that 0.8 is bigger than 0.65, some students demonstrated considerable insight with the responses "0.8 is 0.80", "0.65 is 6 and a half, whereas 0.8 is 8" and "0.8 is different from 0.08". Uarda then gave an explanation using Dienes blocks, taking the flat to represent 1 unit and having the students recognise that a long then represents 0.1 and a cube represents 0.01. Many students appeared to realise that 0.8 is represented by 8 longs and that 0.65 corresponds to 6 longs and 5 cubes, but one student suggested that a cube represented 0.1 and several students called 0.65 “point sixty-five”.

Lesson Discussion

After the lesson, Uarda remarked on how well the students enjoyed the hands-on aspect of the lesson. She was particularly struck by the fact that some students who are normally silent had participated actively: “Monica put up her hand at the back a few times. She’s somebody that generally has no idea. And she was hands-up confidence.”

Students’ tendency to notice irrelevant similarities was noted, and it was decided for the second trial lesson to vary the children’s names and to ask alternately for the larger or smaller number. It was also decided to compare 0.4 and 0.25 in order to make the lollies and measurement tasks more manageable. Bridget left to write the revised scenarios on the whiteboard in her classroom in preparation for her lesson.

Discussion then turned to the question of how general students’ understanding of place value was. It was felt that the approach taken had tended to reinforce whole number thinking (e.g., treating 0.8 as 80 out of 100 rather than 80% of the flat). It was decided to put more emphasis on decimals as fractions of a whole rather than numbers of parts of the whole, even by including questions such as “What is 0.4 of a student’s pony tail?” Ideas were canvassed on how to question students in order to take them beyond the specific insights they had shown, by focussing more strongly on the similarities between the four scenarios. The plan for the repeat lesson (not reported here) was modified accordingly.

As a measure of how much students had learned from the lesson, Uarda suggested a short quiz in which students would be asked to identify the smaller of two decimals; the team then constructed this quiz jointly. Figure 2 shows the resulting quiz.

Circle the smaller number:

(1) 0.8	(2) 0.52	(3) 0.6	(4) 0.8	(5) 1.6534	(6) 2
0.65	0.7	0.298	0.09	1.72	2.2

Figure 2. Short quiz questions.

Assessment

Shortly after the lesson discussion, Uarda administered the quiz to a total of 24 students in her class. About 70% answered Items 1, 2 and 6 correctly, while Items 3, 4 and 5 were only answered correctly by just over 50% of the students. However, five students marked all the *larger* numbers correctly. It could be argued that they understood how to find the larger or smaller of two decimals but had not read the instructions carefully and had proceeded as in the scenarios they had just experienced. On this assumption, the percentages correct would have been about 20% greater than those just quoted.

If this assumption is correct, the result for Item 2 would suggest that the majority of Uarda’s students had generalised their knowledge of 2-place decimals to numbers other than the ones in the given scenarios. However, the zero in Item 4 seems to have introduced difficulties for several students. Most students appeared to be able to cope with numbers with no decimal places, as in Item 6, but it was clear from Items 3 and 4 that more work needed to be done on decimals with more than 2 places.

The assessment results thus confirmed the post-lesson conclusion that the discussion of similarities and differences needed to put greater emphasis on the underlying structure of the decimal notation system.

Discussion

We discuss the results in terms of the four components of the Integrated Model of Teacher Professional Development (Figure 1). Because Application was deliberately avoided in the experimental study, we consider only the first three phases of the Teaching for Abstraction model.

The Domain of Practice: Implementation of the Teaching for Abstraction Model

The lesson certainly incorporated major elements of the Teaching for Abstraction model. Most obvious was the emphasis on exploring a variety of contexts where a concept arises (Phase 1 of the model). This characteristic was probably the result of the team's careful planning, firstly to clearly identify a focus concept or generalisation and secondly to select scenarios that embodied this abstraction and would be familiar to the students.

Similarity recognition (Phase 2) was also present. However, students showed a strong tendency to focus on superficial similarities, and it was difficult for Uarda to bring students' attention to mathematical similarities.

Neither lesson proceeded very far with reification (Phase 3), where the underlying structure could have been laid bare, explained, and formalised. Dienes blocks did not seem to relate well to the similarities that students had observed and were almost a distraction.

The Domain of Consequence: Student Outcomes

The assessment of student outcomes suggested that most students had achieved the narrow objective of ordering one- and two-place decimals. However, understanding was not deep enough to allow many students to extend this knowledge to decimals containing zeros after the decimal point or with more than two places. We attribute this finding to shortcomings in the Similarity and Reification phases of the lesson.

The lesson had some other positive outcomes: The realistic nature of the activities appeared to engage some students more than usual, and there was clear evidence that students were responding well to the chance to think for themselves.

The Personal Domain: Teacher Learning

Uarda emphasised right from the first meeting her belief in the importance of embedding mathematics in realistic, familiar contexts. However, the idea of using more than one context for each concept was clearly new to her, and in the initial discussion sessions the authors had to repeatedly stress the importance of similarity recognition in the Teaching for Abstraction model. Consequently, she put a lot of thought and effort into the process of identifying and selecting appropriate contexts for the trial lesson. Our observations showed that she was skilled in managing a classroom so that students could work on several contexts in one lesson, leaving sufficient time to discuss the students' findings. We conclude that Uarda had successively learned to implement the Familiarity phase of the Teaching for Abstraction model.

Uarda also attempted to implement the Similarity phase in that she challenged students to identify similarities in, and make generalisations from the exploratory activities they had just carried out. However, she did not clearly distinguish superficial and mathematical similarities and did not probe for explanations or forge links to students' existing knowledge. As a result, Uarda did not effectively address the Reification phase. It would seem that she had only just begun to form a concept of the content and purpose of these two phases.

Uarda’s use of Dienes blocks had not been discussed in the lesson planning sessions and represented a return to a more didactic pedagogy. It was an honest attempt to address a known student difficulty, but it only served to disrupt the structure of the Teaching for Abstraction lesson. In this respect, Uarda’s modification was similar to the way teachers in earlier studies had subverted the model by returning to a more familiar lesson structure (White & Mitchelmore, 2010).

The External Domain: Collaborative Lesson Planning

The collaborative lesson planning went according to plan, although it would certainly have been more effective had all the teachers been able to participate to the extent that Uarda did. She certainly had ownership of the Teaching for Abstraction part of her lesson.

However, it is now clear that the professional development we provided for Uarda and her colleagues had several shortcomings. Firstly, we had ignored Dienes’s (1963) dictum that “all variables need to be varied if the full generality of the concept is to be achieved” (p. 158) and had not stressed the importance of varying irrelevant factors when choosing the initial scenarios. However, this principle soon became obvious and was readily corrected in subsequent lessons.

More importantly, we did not stress strongly enough the significance of the crucial steps of similarity recognition and reification. It is clear in hindsight that the teachers needed to spend more time reflecting on mathematical similarities—how to distinguish them from superficial similarities, how to draw them out through careful questioning, how to derive adequate explanations, how to link students’ learning to their previous knowledge, and how to reify similarities to allow for the abstract manipulation of mathematical concepts.

Conclusions and Implications

To answer our research question, it appears that collaborative lesson development of the type we provided did not in this case lead to faithful implementation of the Teaching for Abstraction model. In particular, there were serious difficulties in implementing the Similarity and Reification phases of the model. As a consequence, although students gained from the use of familiar contexts and the challenge of reaching their own conclusions, it seems that they did not reach the desired depth of understanding.

Because most modern curriculum innovations stress the importance of linking abstract concepts to familiar situations, it appears to be relatively easy for teachers to learn the Familiarity phase of the Teaching for Abstraction model. However, the Similarity and Reification phases are unique to the model, and are therefore likely to be completely novel to teachers. Moreover, the whole approach is the reverse of the traditional ABC (Abstract Before Concrete) method of teaching mathematics (Mitchelmore & White, 2000). As Bridget put it, **“**It is the opposite of what we’re doing in school now. It is starting with the blurred and being revealed. Not doing specific instructions first, more a thinking thing.**”** As Sullivan, Clark, and Clark (2009) found in a similar context, teachers may need a lot of help in developing such new skills when they diverge so widely from their normal practice.

We conclude once again that the type of one-off intervention that was the subject of the present study is unlikely to achieve the desired effects. More time is needed for teachers to experiment with the new lesson structure and to reflect on what happens in the classroom and how it affects student learning (see Figure 1). Even if researchers do work collaboratively with teachers to develop potentially more effective Teaching for Abstraction lessons, professional development clearly needs to allow teachers to experience several examples of the application of the model. Only then could we expect them to recognise the underlying structure of the model, reify it into a set of principles that become part of their pedagogical knowledge and beliefs, and apply their understanding to the design of new lessons. In other words, we probably need to teach teachers Teaching for Abstraction by abstraction.

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